If the solution $s(t)$ of the Cauchy problem (4.4) is constructed in an analytical or numerical form, all the kinematic and dynamic characteristics of the motions of the fluid are determined using the constructions in Secs 2 and 3 and formula (4.1).

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THE THERMAL WAKE OF A STREAMLINED BODY†

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The stationary problem of the thermal wake behind a body around which there is a flow of a viscous incompressible fluid is considered within the framework of the full heat-conduction equation. It is assumed that the solution of the corresponding hydrodynamic problem is known. In the case of the hydrodynamic problem, theorems of existence [1, 2] and uniqueness [1] have been proved and the leading term of the expansion [1, 3] at an infinitely remote point has been obtained together with estimates of the remaining terms [1, 4]. Work mainly carried out within the framework of the boundary layer approximation [5] is concerned with the solution of the thermal problem.

1. THE SOLUTION of the hydrodynamic problem can be represented in the form

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_0 + \mathbf{w}(\mathbf{x}), \quad \mathbf{w} = O(1/r), \quad r = |\mathbf{x}| \tag{1.1}$$

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where $\mathbf{v}_0$ is the velocity of the fluid at infinity and the body is assumed to be at rest. In this case, the heat-conduction equation has the form

$$a\Delta T - (\mathbf{v}_0, \nabla T) = (\mathbf{w}, \nabla T) - \frac{\nu}{2c_p}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)^2 \tag{1.2}$$

(a is the thermal diffusivity, c_p is the specific heat capacity at constant pressure and ν is the kinematic viscosity). Equation (1.2) includes a bulk source of heat evolution due to the dissipation of the kinetic energy of the motion of the fluid.

The boundary conditions for Eq. (1.2) are the standard conditions. In particular, we assume that the temperature field $T_*(\mathbf{a})$ is specified on the surface of the body and that the temperature at infinity is equal to zero. By assuming that $\mathbf{w} = O(1/r)$ in the neighbourhood of an infinitely remote point $r = \infty$, Eq. (1.2) can be approximated by the equation

$$a\Delta T_0 - (\mathbf{v}_0, \nabla T) = -q(\mathbf{x}) \tag{1.3}$$

where $q(\mathbf{x}) \in W_2^1$ is a specified heat source, which follows from (1.2). Approximation (1.3) is analogous to the Oseen approximation in the case of the hydrodynamic problem. Its justification follows from the estimates $T = O(1/r)$, $\nabla T = O(1/r^{3/2})$ which are obtained below. It is verified by direct substitution that the fundamental solution of the homogeneous equation (1.3) is

$$G(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi a r} e^{-\lambda s}$$
$$\lambda = |\mathbf{v}_0|/a, \quad s = r - \mathbf{n}_0 \cdot (\mathbf{x}-\mathbf{y}), \quad r = |\mathbf{x}-\mathbf{y}|, \quad \mathbf{n}_0 = \mathbf{v}_0/|\mathbf{v}_0| \tag{1.4}$$

Using Green's function $G(\mathbf{x}-\mathbf{y})$, Eq. (1.3) can be represented in the integral form

$$T_0(\mathbf{x}) = -\int_E G(\mathbf{x}-\mathbf{y})\, q(\mathbf{y})\, d^3y - \oint_\Sigma \left[\left(v_{0j}T_* - a\frac{\partial T_0}{\partial y_j}\right) G(\mathbf{x}-\mathbf{y}) + aT_* \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\right] n_j\, dS \tag{1.5}$$

in which the thermal flux on the surface of the body Σ: $a\partial T_0/\partial n$ is determined as the solution of the corresponding Fredholm integral equations [6] and E is the volume occupied by the fluid.

By using the estimates of the velocity derivatives for the streamline problem [4], it is possible to show that $q(\mathbf{x}) \leqslant c(|\mathbf{x}| + r_0)^{-3}$, where r_0 is the characteristic dimension of the body around which the flow occurs. It follows from this that the volume integral in (1.5) exists for any stationary circumfluence problem.

It is assumed that the surface Σ satisfies the Lyapunov conditions. The integral with respect to an infinitely remote surface is equal to zero. This can be shown using the estimates $G = O(1/r)$, $\nabla G = O(1/r^{3/2})$ which follows from (1.4) and the conditions of the boundedness of the thermal flux at infinity

$$\left|\oint_{S_R} \left[\left(v_{0j}T - a\frac{\partial T}{\partial x_j}\right) n_j\, dS\right| \leqslant C < \infty \tag{1.6}$$

for a sphere S_R of radius R, using the boundary condition $T(\infty) = 0$. The fundamental solution of this problem $G(\mathbf{x})$ also turns out to be useful when investigating the full equations of convective heat transfer (1.2).

On inverting the operator on the left-hand side of (1.2) using $G(\mathbf{x}-\mathbf{y})$, we arrive at the integral equation

$$T(\mathbf{x}) = \int_E T w_j \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\, d^3y - \int_E G(\mathbf{x}-\mathbf{y})\, q(\mathbf{y})\, d^3y -$$

$$-\oint_\Sigma \left[\left(v_j T_* - a\frac{\partial T}{\partial y_j}\right) G(\mathbf{x}-\mathbf{y}) + aT_* \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\right] n_j dS \tag{1.7}$$

A doubly differentiable bounded solution of Eq. (1.7) is sought which possesses a bounded derivative with respect to the normal to the surface of the body. The boundedness of the temperature T_* and the thermal flux $a\partial T/\partial n$ on Σ ensures the existence of the surface integral.

The first volume integral on the right-hand side of (1.7) exists by virtue of the estimates $|w(\mathbf{x})| \leqslant C(|\mathbf{x}|+r_0)^{-3}$ [4], the boundedness of the temperature $|T| \leqslant C$ and the form of $G(\mathbf{x}-\mathbf{y})$ [4]. The second volume integral is identical to the analogous integral in (1.5) and, consequently, also exists. As in the preceding case, the integral over an infinitely remote surface is equal to zero subject to the condition of the boundedness of the total thermal flux on it

$$\varkappa_\infty = \oint_{S_R} \left[\left(v_j T - a\frac{\partial T}{\partial x_j}\right) n_j dS \leqslant C < \infty \quad R \to \infty\right.$$

Equation (1.7) is a Fredholm integral equation of the second kind. Without going into details, it may be assumed that Eq. (1.7) has a generalized solution for each solution of the hydrodynamic problem w_j, q, if the surface of the body Σ satisfies the Lyapunov conditions. The generalized solution T is determined in a similar way to that used in the case of the velocity field in the sense that $(\varphi, LT) = (\varphi, q)$ for each function $\varphi \in C_0^\infty$ and L is the operator of the convective heat-conduction equation $L = (\mathbf{v}, \nabla) - a\Delta$. The proof is essentially the same as the analogous proof for the Oseen equations [1, 2]. In particular, theorems analogous to (2.6) and (2.7) in [1] hold. These assert that, if $q \in C^{r+\alpha}$, $r \geqslant 0$, the solution $T(\mathbf{x}) \in C^{r+2+\alpha}$ and, if the surface of the body $\Sigma \in C^{r+\alpha}$, the temperature on the boundary $T \in C^{1+\alpha}$ and the function q is bounded close to Σ then $\nabla T \in C^{0+\alpha}$ in a closed (external) neighbourhood of Σ and $T(\mathbf{x}) \to T_*$ on Σ.

Let us consider the expansion of the solution (1.7) at an infinitely remote point. The surface integral in (1.7) can be represented in the form of a multiple expansion

$$\Lambda(\mathbf{x}) = \oint_\Sigma \left[\left(v_j T - a\frac{\partial T}{\partial y_j}\right) G(\mathbf{x}-\mathbf{y}) + aT \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\right] n_j dS \tag{1.8}$$

$$\Lambda(\mathbf{x}) = \varkappa G(\mathbf{x}) + \sum_{n=1}^{\infty} \varkappa_{j_1 \ldots j_n} \frac{\partial^n G(\mathbf{x})}{\partial x_{j_1} \ldots \partial x_{j_n}} \tag{1.9}$$

$$\varkappa_{j_1 \ldots j_n} = \oint_\Sigma \frac{(-1)^n}{n!} \left[y_{j_1} \ldots y_{j_n} \left(v_j T - a\frac{\partial T}{\partial y_j}\right) n_j + naT n_{j_1} y_{j_2} \ldots y_{j_n}\right] dS \tag{1.10}$$

$$\varkappa_\infty = \oint_{S_R} \left(v_j T - a\frac{\partial T}{\partial x_j}\right) n_j dS \quad R \to \infty \tag{1.11}$$

The expansion (1.9) is obtained using an expansion of Green's function $G(\mathbf{x}-\mathbf{y})$ in a Taylor's series with respect to $\mathbf{y}$ which is absolutely convergent if the magnitude of $|\mathbf{x}|$ is sufficiently large ($\mathbf{y} \in \Sigma$ which is a bounded surface). Let us now consider the volume integral

$$N(\mathbf{x}) = \int_E T w_j \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\, d^3y \tag{1.12}$$

We will first present the estimate

$$|\nabla G| \leqslant \frac{e^{-\lambda s}}{4\pi a}\left[\frac{1}{r^2}+\frac{\lambda}{r}\sqrt{2\frac{s}{r}}\right] \leqslant \frac{C}{r^{3/2}(1+\lambda s)} \tag{1.13}$$

which is obtained from (1.4) taking account of the relationships

$$|\nabla s|^2 = 2\frac{s}{r}, \quad \sqrt{x}e^{-x} < \frac{1}{1+x}, \quad x \geqslant 0$$

which can be directly verified. The solution of the hydrodynamic problem has the estimate [4]

$$|\mathbf{w}| \leqslant Cr^{-1}(1+\sigma s)^{-1+\varepsilon}, 0 < \varepsilon \leqslant {}^1/_2, \sigma = {}^1/_2 |\mathbf{v}_0|/\nu \tag{1.14}$$

Let us also assume that

$$|T(\mathbf{x})| \leqslant Cr^{-\alpha}, \alpha > {}^1/_2 \tag{1.15}$$

In this case, Assumption 2 from [4] which we shall formulate in the following manner can be applied to (1.12).

The convolution

$$J(\mathbf{x}) = \int\limits_{R^3} W(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, d^3y$$

subject to the condition

$$|f(\mathbf{x})| \leqslant (\sigma r+1)^{-1-\alpha}(\sigma s+1)^{-1+\varepsilon}, \ \alpha > {}^1/_2, \ 0 < \varepsilon \leqslant {}^1/_2$$
$$|W(\mathbf{x})| \leqslant r^{-3/2}(\lambda s+1)^{-1}, \ r = |\mathbf{x}|, \ s = r - \mathbf{n}_0 \cdot \mathbf{x}$$

has the majorant

$$|J(\mathbf{x})| \leqslant Cr^{-\alpha-1/2}[\ln(\sigma r)+1], \ \sigma r \geqslant 1$$

Using this assertion from (1.12) and taking account of the estimates (1.13)–(1.15), we get

$$|N(\mathbf{x})| \leqslant C_1 r^{-\alpha-1/2}[\ln(\sigma r)+1] \tag{1.16}$$

It is seen from expansion (1.9) and expression (1.6) that the surface integral

$$\Lambda(\mathbf{x}) = O(1/r) \tag{1.17}$$

It follows from (1.16) that the volume integral $N(\mathbf{x})$ is negligibly small when $r \to \infty$ with respect to the surface integral $\Lambda(\mathbf{x})$. Taking account of expansion (1.9) in the case when there is no bulk thermal source ($q(\mathbf{x}) \equiv 0$), we write the asymptotic form for the temperature in the form

$$T(\mathbf{x}) = -\varkappa G(\mathbf{x}) + \theta(\mathbf{x}), \ \theta(\mathbf{x}) = O(r^{-3/2}\ln(\sigma r)) \tag{1.18}$$

since, in accordance with (1.17), the exponent α in estimate (1.15) is equal to unity which, in turn, follows from (1.7) taking account of (1.8), (1.12), (1.16) and (1.17).

We note that condition (1.15) can be obtained, using the methods in [7], from the condition of the boundedness of the Dirichlet integral for the temperature (see the Appendix)

$$\int\limits_E (\nabla T)^2 d^3x < \infty \tag{1.19}$$

This condition has a clear physical meaning: relationship (1.19) in conjunction with the boundedness of the energy dissipation ensures the boundedness of the entropy production [8].

If account is taken of the bulk heat evolution $q(\mathbf{x})$ in the form of the dissipation of the kinetic energy of the fluid, the asymptotic representation (1.18) does not undergo any appreciable changes

$$T(\mathbf{x}) = -(\varkappa + D)\, G(\mathbf{x}) + \theta(\mathbf{x}), \quad \theta(\mathbf{x}) = O(r^{-3/2} \ln(\sigma r))$$
$$D = \int_E q(\mathbf{x})\, d^3x = \frac{v}{2c_p} \int_E \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 d^3x < \infty \tag{1.20}$$

Expression (1.20) can be proved by applying the corresponding estimates for $\partial\omega/\partial x$ and $G(\mathbf{x}-\mathbf{y}) - G(\mathbf{x})$ and using Assumption 2 of [4]. It can be shown from the heat-conduction equation (1.2) that

$$\varkappa + D = \varkappa_\infty = \oint_{S_R} \left(v_j T - a \frac{\partial T}{\partial x_j}\right) n_j dS, \quad R \to \infty \tag{1.21}$$

Hence, the leading term in the temperature expansion (1.20) is determined by an exact conservation integral, that is, by the total heat flux at infinity and, at the same time, according to (1.4) and (1.20)

$$T(\mathbf{x}) = O(1/r) \tag{1.22}$$

2. Let us now consider a turbulent thermal wake. We shall start off from the averaged convective transport equations

$$\alpha \Delta \bar{T} - (\mathbf{v}_0, \nabla \bar{T}) = (\mathbf{u}, \nabla \bar{T}) + \overline{(\mathbf{w}', \nabla T')} - Q(\mathbf{x})$$
$$\mathbf{u}(\mathbf{x}) = \overline{\mathbf{w}(\mathbf{x}, t)}, \quad \mathbf{w} = \mathbf{u} + \mathbf{w}', \quad T = \bar{T} + T' \tag{2.1}$$
$$Q(\mathbf{x}) = \frac{\gamma}{2c_p} \overline{\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right)^2}$$

Equation (2.1) differs from (1.2) in that it contains an additional "heat source" $(\mathbf{w}', \nabla T')$ which arises on account of the presence of turbulent thermal stresses. These stresses can be obtained from supplementary equations in the pulsations of w_i' and T'. However, in order to derive the asymptotic estimates, it suffices to confine ourselves to certain general assumptions regarding the asymptotic behaviour of $w_i'T'$.

On inverting the linear operator on the left-hand side of (2.1) using the fundamental solution $G(\mathbf{x}-\mathbf{y})$ (1.4), we arrive at the equation

$$\bar{T}(\mathbf{x}) = \int_E \overline{Tw_j} \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\, d^3y - \int_E G(\mathbf{x}-\mathbf{y})\, Q(\mathbf{y})\, d^3y -$$
$$- \oint_\Sigma \left[\left(\overline{v_j T_*} - a \frac{\partial \bar{T}}{\partial y_j}\right) G(\mathbf{x}-\mathbf{y}) + a\bar{T} \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\right] n_j dS \tag{2.2}$$
$$\overline{Tw_j} = \bar{T} u_j + \overline{T'w_j'}$$

The existance of the integrals in Eq. (2.2) is doubtful in view of the unknown asymptotic behaviour of the function $T'w_j'$. However, it is well known [9] that a spatial wake becomes laminar at a distance from the body and, moreover, that this occurs at a finite but, perhaps, fairly large distance. By virtue of the assumed boundedness of the field quantities occurring in (2.2), this means that one can use the estimates for laminar flow (Sec. 1) in the upper limit in the volume integrals and, consequently, it may be assumed that these integrals also exist in the case of a turbulent flow behind a uniformly moving body.

Confining ourselves to these qualitative arguments (a more rigorous proof requires the solution of the problem of stability with respect to arbitrary spatial perturbations of the laminar hydrodynamic and thermal wake), let us consider the volume integral

$$I(\mathbf{x}, t) = \int_E T w_j \frac{\partial}{\partial y_j} G(\mathbf{x}-\mathbf{y})\, d^3y \tag{2.3}$$

By using the scheme for the proof of the asymptotic behaviour of a hydrodynamic wake in [7], in which time is considered as a parameter, from the condition for the boundedness of Dirichlet type integrals

$$\int_E (\nabla T)^2 d^3x \leqslant C_1, \quad \int_E \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 d^3x \leqslant C_2$$
$$C_1, C_2 = \text{const} < \infty \tag{2.4}$$

it is possible to obtain the estimate (see the Appendix)

$$| I(x, t) | \leqslant C_3 r^{-\alpha}, \ \alpha \geqslant {}^2/_3, \ C_3 = \text{const} < \infty \tag{2.5}$$

Conditions (2.4) are the natural physical requirement of the boundedness of entropy production in the case of turbulent motion in a wake. From (2.5), we get

$$| I(\mathbf{x}) | \leqslant \overline{| I(\mathbf{x}, t) |} \leqslant C_3 r^{-\alpha}, \ I(\mathbf{x}) = \overline{I(\mathbf{x}, t)}, \ \alpha \geqslant {}^2/_3 \tag{2.6}$$

The contribution to the volume integral from a source $Q(\mathbf{x})$ together with the surface integral in (2.2) can be estimated in the same way as in Sec. 1, to be $O(1/r)$. Hence, we have the estimate

$$| T(\mathbf{x}) | \leqslant C r^{-\alpha}, \ 1 \geqslant \alpha \geqslant {}^2/_3, \ C = \text{const} \tag{2.7}$$

Simple evaluation considerations for a turbulent thermal wake yield $\alpha = 2/3$. The result follows from the condition that the convective thermal flux is specified as $J_Q \approx \rho u_0 T\delta^2 = \text{const}$ and the estimate $d\delta/dz \approx T/T_0$ for the width of the thermal wake δ which propagates along the z-axis, when $\delta \approx z^{1/3}$, $T \approx z^{-2/3}$.

The occurrence of a memory of the shape of a streamlined body with a hydrodynamic turbulent self-similar wake was pointed out in the experimental papers [10, 11]. It follows from what has been said above that recollection of shape can only hold when $\alpha \leqslant 1$ in (2.6) and (2.7). If $\alpha > 1$ in (2.5) and (2.6), then the self-similar turbulent wake will be determined by the total heat flux as was pointed out in Sec. 1. We note that, in the domain of the limiting asymptotic behaviour of the distant wake, memory of shape must not be a consequence of the flow becoming laminar [9] and the results of Sec. 1 for a laminar flow, according to which the leading term of the expansion is determined by a single conservation integral, that is, by the total thermal flux.

Next, by using Assumption 2 of [4] which has been formulated in Sec. 1, it is easy to investigate under which conditions memory of the shape of a body, around which a flow occurs with a turbulent thermal wake, can arise.

Let the estimate

$$| \overline{T w_j} | \leqslant C r^{-1-\beta}, \ \sigma > {}^1/_2, \ C = \text{const} \tag{2.8}$$

hold.

Then,

$$| I(\mathbf{x}) | \leqslant Cr^{-\beta-1/2} [\ln(\sigma r) + 1] \tag{2.9}$$

It is obvious from this that the condition $\beta > 1/2$ precludes the possibility of a shape memory since the leading term in the expansion is determined by a surface integral of the order of $1/r$ and depends solely on the total thermal flux. An additional necessary condition for the possibility of a memory of the shape of a body around which a flow with a turbulent thermal wake occurs can be obtained from relationships (2.8) and (2.9) in the form

$$| \overline{Tw_j} | r^{3/2} \geqslant C > 0 \quad (\beta \leqslant 1/2) \tag{2.10}$$

which represents the need for a sufficiently slow decay (not faster than $r^{-3/2}$) of the turbulent thermal stresses in the domain of intermediate asymptotic behaviour of the self-similar turbulent thermal wake. Relationships (2.10) can be used in the experimental investigation of the possibility of the occurrence of a memory of the shape of a body around which a flow occurs in the case of a turbulent thermal wake.

APPENDIX

We will show, following [7], that, if the temperature field belongs to the Dirichlet class (1.19), the solution of the stationary thermal problem has the estimate (1.15). The proof in the case of the hydrodynamic problem [7] can almost be transferred word-for-word to the thermal problem and we shall therefore confine ourselves to the formulation of the corresponding theorems while indicating the necessary changes which are however unimportant for the proof.

Consider the operator

$$h_m(\varphi) = \int\limits_{R^3} \frac{\partial}{\partial y_m} G(x-y)\,\varphi(y)\, d^3y$$

in the space $L^r(R^3)$. We shall subsequently use the notation [7]

$$|f|_{D,r} = \Big(\int\limits_D |f(\mathbf{x})|^r d^3x\Big)^{1/r}$$

and, if $d = R^3$ or $D = E$, simply write $|f_r|$. The x_1-axis is directed along the circumfluence velocity $\mathbf{v}_0$.

Proposition 1. If $m = 1$, then h_m is an operator from L^r into L^r and

$$| h_1(\varphi) |_r \leqslant A_r | \varphi_r |, \quad 1 < r < 4$$

This assertion, as well as its proof, are completely analogous to supposition 1 from [7]. It should also be noted that the Fourier transformation of $\partial G/\partial x_m$

$$g_m(\mathbf{u}) = \int\limits_{R^3} \frac{\partial G(\mathbf{x})}{\partial x_m} e^{i\mathbf{x}\cdot\mathbf{u}} d^3x$$

has the form

$$g_m(\mathbf{u}) = iu_m / (u^2 - 2i\lambda u_1)$$

whence the required assertion follows using the results of Mikhlin and Lizorkin cited in [7].

The following assertions [7] are not subject to any changes.

Proposition 2. (A special case of Sobolev's theorem.) Let the function $\varphi(\mathbf{x}) \in C(R^3)$ be absolutely continuous with respect to each of the variables subject to the condition that the remaining variables take arbitrary fixed values and let it have a finite Dirichlet integral $|\nabla\varphi|_2 < \infty$. Let us further assume that $\varphi(\mathbf{x}) \to 0$ when $|\mathbf{x}| \to \infty$. The inequality $|\varphi|_6 \leq A|\nabla\varphi|_2$ then holds, where A is an absolute constant.

Proposition 3. Let φ satisfy the conditions of Proposition 2 and, furthermore, suppose $\varphi_{x_1}|_r < \infty$, where $1 < r < 2$.

Then

$$|\varphi_{3r}| \leqslant B[|\varphi_{x_1}|_r |\varphi_{x_2}|_r |\varphi_{x_3}|_r],\ B = [3r^2(3r-2)]^{1/3}$$

Using Proposition 2 and the possibility of a finite continuation of the temperature field in the neighbourhood of the surface Σ [the temperature field within the body can, for example, be found by solving the Laplace equation with the corresponding boundary conditions for $T_*(\mathbf{s})$], we find $|T|_6 < \infty$.

Proposition 4. The relationship $|T|_4 < \infty$ holds.

This assertion is completely analogous to Proposition 4 of [7] ($|\mathbf{v}|_4 < \infty$). The proof is based on an analysis of (1.7). The derivatives with respect to x_1 are investigated separately for the volume and surface integrals on the right-hand side of (1.7). It can be directly verified that all of the relationships in the proof of [7] remain valid on making the substitutions $H_{ij} \to G$ and $\mathbf{v} \to T$ whereupon Proposition 4 follows.

Proposition 5 of [7] refers to the integrability of the pressure and is not used in our case. Three propositions follow next in [7] which are directed towards the proof of the assertion that $|\mathbf{v}|_r < \infty$ if $r < 4$ but is fairly close to four. Auxiliary constructions [7] are used for this and, in particular, a truncated fundamental solution is introduced which differs from zero in the immediate neighbourhood of the wake. This part of the paper contains quite long proofs, the reproduction of which, while making small changes, is hardly sensible. We merely note that the proofs substantially rest upon an integral equation of the type of (1.7). By comparing Eq. (1.7) with the corresponding equations (1.15) and (2.9) of [7], the great similarity between them can be noted and it can be shown that, in the case being considered, the scheme of the proof due to Babenko [7] remains valid and, here, it suffices to make the substitutions $\mathbf{H}_{ij} \to G$ and $\mathbf{v} \to T$. This is founded in the identical functional properties of the fundamental solutions H_{ij} and G, by their identical asymptotic behaviour, together with the first-order derivatives and the identical structure of the corresponding integral equations.

Proposition 9. The inequality $|T|_r < \infty$ holds for all $r > 2$.

Proposition 10. Let

$$\varphi(\xi) = \max_{|\mathbf{x}| \geqslant \xi} |T(\mathbf{x})|,\ \xi \geqslant 1$$

There then exists a $\xi^* > 0$ such that

$$\varphi(\xi) < (8/\xi)^{1/\beta} \text{ for } \xi > \xi^*,\ \beta = 3/2 - \varepsilon,\ 0 < \varepsilon \ll 1.$$

The proof is a word-for-word repeat of the proof of propositions 9 and 10 of [7] with the substitution $\mathbf{v} \to T$. The estimate

$$|T(\mathbf{x})| \leqslant C|\mathbf{x}|^{-\alpha},\ \alpha \geqslant 2/3$$

can be obtained from this, and inequality (1.15) is thereby satisfied.

Since Babenko's proof [7] rests solely on a series of estimates of integral operators and only makes use of fairly general properties of the solution being estimated, his scheme can be applied to a broad class of problems where there are analogous operators. In particular, apart from the stationary thermal problem of the flow round a body, it is possible to treat turbulent thermal and hydrodynamic wakes. It would be expected that estimates of integral operators having the same fundamental solution as kernels would not essentially change. Here, it is convenient to carry out time averaging or averaging over an ensemble in the estimates which have already been made where time is treated as a parameter. By successively carrying out this reasoning, we arrive at an estimate of the mean temperature in the form of (2.7) which is also valid in the case of a laminar trail but, unlike in the laminar case, proposition 2 of [4] can no longer be used here. This is associated with the fact that the thermal stresses $w_j'T'$, just like the Reynolds stresses $w_i'w_j'$, cannot decay according to the required law $\approx r^{-\alpha}$, $\alpha > 3/2$. Hence, in the turbulent domain of a distant wake, the contribution of the volume integral to (2.2) cannot be small. We cannot rule out the possibility that the presence of a memory of the shape of the body around which the flow occurs in the case of a hydrodynamic trail is, in fact, associated with this.

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